Optimal portfolio choice with path dependent labor income

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IMS-FIPS Workshop, London
September 10, 2018

Outline

Overview

- Lifecycle portfolio choice problem with borrowing (state) constraints where an agent receives labor income.
- Novelty: *path-dependency of the wage income process* ("slow" adjustment to financial market shocks; "learning" your income) which leads to an *infinite dimensional* stochastic optimal control problem.
- We solve completely the problem, and find explicitly the optimal controls in feedback form. Tool: explicit solution to the associated infinite dimensional Hamilton-Jacobi-Bellman (HJB) equation.
- First step towards more general and interesting problems and more general solution methods.

Motivation: Portfolio choice

- Merton (1971): lifetime investment in risky stocks and riskless asset. Optimal for agents to allocate a *constant fraction of wealth in the risky asset* throughout their lives.
- Importance of labor income in shaping portfolio choice: e.g., Bodie et al. (1992), Campbell-Viceira (2002), Fahri-Panageas (2007), Dybvig-Liu (2010). The **total wealth** of an agent is given by both financial wealth and **human capital**, i.e., the market value of future labor income.
- Key finding I: investors should allocate a constant fraction of their **total wealth** to the risky asset.
- Key finding II: negative **hedging demand** for risky assets arises from the implicit holding of the risky assets in human capital.

Motivation: Human Capital I

Labour income dynamics

- ARMA processes commonly used to model the stochastic component of wages (e.g., MaCurdy, 1982; Abowd-Card, 1989; Meghir-Pistaferri, 2004; Storesletten et al., 2004).
- Stochastic Delay Differential Equations (SDDEs) as natural continuous time counterparts of ARMA processes: Reiss (2002), Lorenz (2006), Dunsmuir et al. (2016).

Sticky wages

- Empirical evidence on wage rigidity suggests that labor income adjusts slowly to financial market shocks (e.g., Khan, 1997; Dickens et a., 2007; LeBihan et al., 2012).
- Delayed labor income dynamics as a tractable model to capture this feature.

Motivation: Human Capital II

Learning your income

- Shocks in labor income have modest persistency when heterogeneity in income growth rates is taken into account.
- Allowing agents to learn in (say) a Bayesian way about income growth can match several empirical features of consumption data (e.g., Guvenen, 2007, 2009).
- Bounded rationality and rational inattention can support the use of moving averages instead of optimal filters (e.g., Zhu and Zhou, 2009).
- Path dependent labor income retains tractability and delivers explicit solutions.

Outline

The model of Dybvig and Liu (2010)

Financial market of Black & Scholes type:

$$
\begin{aligned}
dS_0(t) =& rS_0(t)dt \\
dS_1(t) =& S_1(t)\mu dt + S_1(t)\sigma dZ(t),
\end{aligned}
$$

with $0 < r < \mu$, $\sigma > 0$.

- Z is a Wiener process on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
- We consider one risky asset for illustration only, the case of $n > 1$ risky assets working in a similar way.

Consider the state equation (budget constraint and wage process)

$$
\begin{cases}
dW(t) = & \big[W(t)r + \theta(t)(\mu - r) - c(t) - \delta\big(B(t) - W(t)\big)\big]\, dt \\
& +(1 - R(t))y(t)dt + \theta(t)\sigma dZ(t), \quad W(0) = W_0 \\
dy(t) = & y(t)\big(\mu_y dt + \sigma_y dZ(t)\big), \quad y(0) = y_0
\end{cases}
$$

- $W(t)$ wealth process (state)
- $y(t)$ labor income process (state)
- $\theta(t)$ investment in the risky asset (control)
- $c(t)$ consumption (control)
- $B(t)$ bequest (control)
- $R(t) := \mathbb{I}_{\{T \le t\}}$ and T is the retirement date (control)
- $\delta > 0$ constant rate of mortality
- $\mu_y, \sigma_y > 0$.

- The agent's death time τ_δ is modeled as a Poisson arrival time (with parameter $\delta > 0$) independent of the Wiener process Z
- We should consider as reference filtration the one generated by τ_δ and Z, but we will actually work on $\{\tau_\delta > t\}$.
- $B(t)$ is the bequest the agent targets for his/her beneficiaries:
    - for $W(t) - B(t) < 0$, the agent purchases continously life insurance with premium flow $\delta(B(t) - W(t))$;
    - for $W(t) - B(t) > 0$, the agent is essentially receiving a life annuity flow $\delta(B(t) - W(t))$, as (s)he trades wealth in the event of death for a cash inflow while living.

- **Goal**: maximize over $\big(c(\cdot), B(\cdot), \theta(\cdot), T\big)$ the objective

$$\mathbb{E}\left\{\int_0^{\tau_\delta} e^{-\rho t}\left((1-R(t))\frac{c(t)^{1-\gamma}}{1-\gamma}+R(t)\frac{(Kc(t))^{1-\gamma}}{1-\gamma}\right)dt \right. \\ \left. + e^{-\rho\tau_\delta}\frac{\big(kB(\tau_\delta)\big)^{1-\gamma}}{1-\gamma}dt\right\},$$

  where $K>1$ allows the utility from consumption to differ before and after T, and $k>0$ measures the intensity of preference for leaving a bequest.

- The expectation above can be written as follows:

$$J(W_0, y_0; c, B, \theta, T) := \mathbb{E}\left\{\int_0^{+\infty} e^{-(\rho+\delta)t}\left(\frac{(K^{R(t)}c(t))^{1-\gamma}}{1-\gamma} \right.\right. \\ \left.\left. + \delta\frac{\big(kB(t)\big)^{1-\gamma}}{1-\gamma}\right)dt\right\}$$

The state constraint

Dybvig-Liu (2010), Problem 1

For fixed retirement date $T \leq +\infty$, consider the following no-borrowing-without-repayment constraint:

$$W(t) \geq -g(t)y(t),$$

with

$$g(t) := \left(\frac{1 - e^{-\beta_1 (T-t)}}{\beta_1} \right)^+,$$

where we assume $\beta_1 > 0$, with $\beta_1 := r + \delta - \mu_y + \frac{(\mu - r)}{\sigma}\sigma_y$.

Meaning of the constraint

Let $\xi(t)$ be the mortality risk adjusted state price density:

$$\xi(t) := e^{-(r+\delta+\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2})t-\frac{(\mu-r)}{\sigma}Z(t)},$$

i.e., the solution of

$$\begin{cases} d\xi(t) & = -\xi(t)(r+\delta)dt - \xi(t)\frac{\mu-r}{\sigma}dZ(t), \\ \xi(0) & = 1. \end{cases}$$

Then

$$g(t)y(t) = \xi(t)^{-1}\mathbb{E}\left(\int_t^T y(s)\xi(s)ds\middle|\mathcal{F}_t\right),$$

which is nothing else than the *human capital* at time t.

Outline

1. Overview and motivation
2. Benchmark model (no path dependency)
3. Path-dependent wages
4. Conclusion

Our model

- For simplicity we focus on the infinite horizon case ($T = +\infty$).
- State equation:

$$
\begin{aligned}
dW(t) &= \big[W(t)r + \theta(t)(\mu - r) - c(t) - \delta\big(B(t) - W(t)\big)\big]\, dt \\
&\quad + y(t)dt + \theta(t)\sigma dZ(t), \quad W(0) = W_0 \\
dy(t) &= \left(y(t)\mu_y + \int_{-d}^{0} \alpha(\eta) y(t+\eta) d\eta\right) dt + y(t)\sigma_y dZ(t), \\
y(0) &= y_0, \quad y(\eta) = y_1(\eta) \quad \forall \eta \in [-d, 0).
\end{aligned}
$$

- $W(t)$, $y(t)$, $\theta(t)$, $c(t)$, $B(t)$, as before.
- $\alpha(\cdot)$ square integrable function.

$$J_1(W_0, y_0, y_1; c, B, \theta) :=$$

$$\mathbb{E}\left\{\int_0^{+\infty} e^{-(\rho+\delta)t}\left(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta\frac{\left(kB(t)\right)^{1-\gamma}}{1-\gamma}\right) dt\right\}. \tag{1}$$

Problem

Given $T = +\infty$, choose $c(\cdot)$, $\theta(\cdot)$, $B(\cdot)$ to maximize (1), with the following no-borrowing-without-repayment constraint:

$$W(t) \geq -\left(Gy(t) + \int_{-d}^{0} H(\eta)y(t+\eta)d\eta\right).$$

- After some work we can write (Biffis-Prosdocimi-Goldys, 2015):

$$\xi(t)^{-1}E\Big(\int_t^{+\infty} y(s)\xi(s)ds\Big|\mathcal{F}_t\Big) = Gy(t) + \int_{-d}^{0} H(\eta)y(t+\eta)d\eta.$$

- The constant G and the function H are given by $G := (\beta_1 - \beta_\infty)^{-1}$, $H(\eta) := \int_{-d}^{\eta} e^{-(r+\delta)(\eta-s)}\alpha(s)ds$, with $\beta_\infty := \int_{-d}^{0} e^{-(r+\delta)s}\alpha(s)ds$.
- For $\alpha = 0$ we have $H = 0$ and G coincides with g.
- The above shows that **human capital** is now shaped by two components:
  - Current market value of the past trajectory of labor income, $\int_{-d}^{0} H(\eta)y(t+\eta)d\eta$.
  - Current market value of the future labor income stream, $Gy(t)$.

Stochastic control problem, infinite horizon I

- State space H, Hilbert space. Control space C complete metric space.
- State equation

$$\begin{cases} dx(t) & = b\big(x(t), c(t)\big)dt + \sigma\big(x(t), c(t)\big)dZ(t) \\ x(s) & = y, \qquad s \geq 0,\ y \in H \end{cases}$$

- Set of admissible controls (here when C is bounded, if not integrability properties are needed)

$$\mathcal{U} := \{c : [0, +\infty) \times \Omega \longrightarrow C \mid c \text{ is } \mathcal{F}_t\text{-adapted}\}.$$

- Objective functional

$$J\big(s, y; c(\cdot)\big) := \mathbb{E}\Big\{ \int_s^{+\infty} e^{-\rho t} f\big(x^{(s,y)}(t), c(t)\big)\, dt \Big\},$$

Stochastic control problem, infinite horizon 2

- value function

$$V(s,y) := \sup_{c(\cdot)\in\mathcal{U}^s} J\big(s,y;c(\cdot)\big), \text{ for any } (s,y)\in[0,+\infty)\times\mathbb{R}$$

  we have

$$V(s,y) = e^{-\rho s}V(0,y) = e^{-\rho s}V_0(y).$$

- Hamilton-Jacobi-Bellman equation for V_0

$$\rho v \quad = \mathcal{H}\big(x, v_x, v_{xx}\big) \text{ for any } y\in\mathbb{R}$$

  where

$$\mathcal{H}\big(x,p,P\big) = \sup_{c\in C}\{f(x,c) + b(x,c)p + \frac{1}{2}\sigma^2(x,c)P\}$$

Delay equations as ODEs in infinite dimensional spaces

- The state equation of $y(\cdot)$ is a stochastic delay differential equation.
- Classical theory works for Markovian state equations.
- We reformulate the problem in an infinite dimensional Hilbert space (e.g., Vinter, 1975; Chojnowska-Michalik, 1978; Da Prato-Zabczyk, 2014; Fabbri-Gozzi-Swiech, 2017).
- Consider the Hilbert space

$$\mathcal{H} := \mathbb{R} \times L^2\big([-d, 0]; \mathbb{R}\big),$$

with inner product for $x = (x_0, x_1), z = (z_0, z_1) \in \mathcal{H}$

$$\langle x, z\rangle_{\mathcal{H}} := x_0 z_0 + \int_{-d}^{0} x_1(\xi) z_1(\xi) d\xi$$

$$= x_0 z_0 + \langle x_1, z_1\rangle_{L^2}$$

- Set
$$X(t) = \big(X_0(t), X_1(t)\big) := \big(y(t), y(t+\xi)_{|\xi\in[-d,0]}\big),$$
- $X(t)$ is an element of $\mathcal{H}$ for all $t \in [0, +\infty)$.
- Let X satisfy
$$dX(t) = AX(t)dt + CX(t)dZ(t), \quad X(0) = (y_0, y_1) \in \mathcal{H}$$
with
$$\begin{aligned} A(x_0, x_1) &:= \big(\mu_y x_0 + \langle\alpha(\cdot), x_1(\cdot)\rangle_{L^2}, x_1'(\cdot)\big), \\ C(x_0, x_1) &:= (x_0\sigma_y, 0) \end{aligned}$$
- Then, the original problem is equivalent to the control problem with state X in the infinite dimensional space $\mathcal{H}$ (e.g., Chojnowska 1989, Gozzi-Marinelli, 2004).

Results

Theorem

The value function V_0 *is*

$$V_0(W, x_0, x_1) := f_\infty^\gamma \frac{\Gamma^{1-\gamma}}{1-\gamma},$$

where

$$f_\infty := (1 + \delta k^{\frac{1}{\gamma}-1})\nu,$$

$$\nu := \frac{\gamma}{\rho + \delta - (1-\gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma})} > 0.$$

$$\Gamma := W_0 + Gx_0 + \langle H, x_1 \rangle_{L^2} \geq 0,$$

- The optimal strategies are given by:

$$c^*(t) := f_\infty^{-1}\Gamma^*(t)$$
$$B^*(t) := k^{-b} f_\infty^{-1}\Gamma^*(t)$$
$$\theta^*(t) := \frac{(\mu - r)\Gamma^*(t)}{\gamma\sigma^2} - \frac{\sigma_y}{\sigma} G y(t),$$

where $\Gamma^*(t) := W^*(t) + GX_0(t) + \langle H, X_1(t,\cdot)\rangle_{L^2}$.

- We have

$$\frac{d\Gamma^*(t)}{\Gamma^*(t)} = \Big[r + \delta + \frac{1}{\gamma}\big(\frac{\mu - r}{\sigma}\big)^2 - f_\infty^{-1}\big(1 + \delta k^{-b}\big)\Big] dt + \frac{\mu - r}{\gamma\sigma} dZ(t).$$

Discussion

- With no labor income risk ($\sigma_y = 0$), the optimal ratios $\frac{\theta^*}{\Gamma^*}$ and $\frac{c^*}{\Gamma^*}$ are constant, as in the Merton model.
- Taking $\alpha = 0$, we recover the results of Dybvig-Liu.
- With $\alpha \neq 0$, the same logic as in Dybvig-Liu applies, but optimal total wealth (financial wealth + human capital) is now given by Γ^*:

$$\Gamma^*(t) = W^*(t) + GX_0(t) + \langle H(t,\cdot), X_1(t,\cdot)\rangle_{L^2}.$$

- The ratio $\frac{\theta^*}{\Gamma^*}$ is no longer constant and the negative hedging demand term $\frac{\sigma_y}{\sigma} Gy(t)$ only takes into account the 'future component' of human capital.
- Richer empirical predictions than in the standard case: portfolio choice (e.g., stock market participation) depends on the relative importance of the past vs. future component of human capital.

Sketch of the proof

- Guess the value function to be

$$V(W_0, x_0, x_1) := f_\infty^\gamma \frac{(W_0 + Gx_0 + \langle H, x_1 \rangle_{L^2})^{1-\gamma}}{1-\gamma}.$$

- Putting V in the HJB equation, gives equations for f, G, H.
- Solving these equations, we get that f, G, H are the constant as defined before.
- V is $\mathcal{C}^{1,2}$.
- Verification Theorem holds and the optimal feedback strategies are admissible.

Remarks I

Total wealth zero boundary:

- The borrowing constraint is not always slack.
- The case of binding constraint is reduced to a problem of viability.
- As opposed to Merton-type problems, the agent is not fully invested in the riskless asset along the boundary.
- At the zero boundary we have $c = 0$, $B = 0$, and $\theta = -\frac{\sigma_y}{\sigma} Gy(t)$.
- The agent is still invested in the risky asset, as (s)he needs to fully hedge his/her labor income risk.

Remarks

Verification and preference parameter $\gamma > 0$:

- We cover in detail both the case of $\gamma \in (0, 1)$ and $\gamma > 1$.
- The first case is standard.
- The second case is not: it is at best neglected in the literature. We address this case and prove it explicitly.

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Conclusion and further/future research

Summary

- Extension of Merton's problem to the case of realistic labor income dynamics and constraints.
- Explicit solutions can better match empirical data (e.g., hump shaped risky asset allocations, cross-sectional heterogeneity of portfolio choices, etc.).

Extensions

- The case with given retirement date (finite horizon) or with linear path dependent diffusion coefficient can be solved in a similar way.
- More general problems (e.g. non linear equation for y) call for new theoretical results on HJB equations or on the use of alternative methods (BSDEs through randomization, Maximum Principle, etc.).
[Lines of research: regularization of viscosity solutions using the classical definition (Fabbri-Gozzi-Swiech), or the PPDE definition (Ekren-Touzi-Zhang) in the finite dimensional case, and CossoFedericoGozziRosestolato-Touzi in the infinite dimensional case.]

THANK YOU